FORMULAS OF THE PERSPECTIVE CARTOGRAPHIC PROJECTION FOR PLANETS AND ASTEROIDS OF ARBITRARY SHAPE. E. V. Shalygin, Yu. I. Velikodsky, and V. V. Korokhin. Kharkov Astronomical Observatory. Sumskaya Ul., 35, Kharkov, 61022, Ukraine. E-mail: evgen@astron.kharkov.ua

Abstract: Formulas of transformation between coordinates on image plane, planetocentric coordinates and photometric conditions of observation for arbitrary planet have been obtained. An example with ellipsoidal planet has been considered.

Introduction: When processing planetary images it is necessary to transform image coordinates to planetocentric coordinates or back. Also at photometric studies it is necessary to calculate photometric conditions (geometry) of observation for each point on the planet surface.

Often at realization of such transformations it is supposed that the planet image is in orthographic projection. But it is true only if ratio of the planet size to distance to it (i.e. the angular size of the planet) is negligible for accuracy of the task being solved. Otherwise it is necessary to suppose that the planet image is in the perspective projection (this is especially appreciable at the observations from the board of space vehicles approaching with planets and asteroids).

Formulas of the perspective projection are often obtained for a spherical planet or for some special cases of nonspherical planets. We have tried to make a step to deriving formulas of coordinate transformation in most general form: let us consider images of planets of arbitrary shape in the perspective azimuthal projection. Let us suppose that it is possible to set the form of a planet with equation F(x,y,z)=0 or a set of such equations for different parts of the surface. Let us suppose also, that the line of sight is not directed strongly to the center of the planet, i.e. we deal with more general Tilted Perspective Projection rather than Vertical Perspective Projection.

Problem definition: Let us choose a system of rectangular coordinates (XYZ) in which it is more suitable to set the shape of the planet surface with a such equation:

$$F(x,y,z)=0. \tag{1}$$

If the planet is an ellipsoid, the equation (1) can be written like this:

$$x^2/A^2 + y^2/B^2 + z^2/C^2 = 1, \tag{2}$$

where A, B, C – ellipsoid semi-axes.

Let us name the center of system (XYZ) "the center of the planet".

Let us choose a plane of the perspective projection such that the line of sight crosses it at right angles in a point, placed on the distance D from the observer, where D - distance between the observer and the center of the planet. In this plane we shall consider the image (projection) of a planet, that is equivalent to the image on a photodetector (as a rule, it is a rectangular CCD-image). Let us set coordinates on the image (on the projection plane) x_P and y_P (let axis X_P be directed to the right, and Y_P – upward).

Let us introduce an additional rectangular system of coordinates (X′Y′Z′), with the center in the point of crossing of the line of sight with the projection plane (here axis Z′ is directed to the observer, and axes X′ and Y′ coincide with axes X_P and Y_P correspondingly). In this system the observer has coordinates $x'=y'=0$, $z'=D$. Let us note that at $D \to \infty$ the perspective projection approaches to orthographic one.

Let coordinates in all systems be measured in the same units – for example, in image pixels. Converting to other units (to kilometers, angular seconds) can be performed, knowing scale of the image and distance to the planet.

Since we use the Tilted Perspective Projection, let us consider, that the center of the planet is displaced relative to the direction of the line of sight on an angle ρ (on the coelosphere) in a direction with azimuth ψ counted in the image plane from the positive direction of axis Y′ anticlockwise. If Vertical Perspective is enough, in all formulas it is possible to set $\rho=\psi=0$.

Let us consider the system of planetocentric coordinates: b – a latitude, l – a longitude. Let us define it as spherical system of coordinates with center in the center of the planet, the equator of which places in plane XOZ, axis Y is directed to a north pole, and axis Z is directed to the point with coordinates b=l=0.

And now let us derive the formulas of transformation from coordinates on the image plane (x_P,y_P) to planetocentric coordinates (b,l), and also inverse transformation and formulas of calculation of the observational photometric conditions.

Transformation $(x_P,y_P) \to (b,l)$: Coordinates on the CCD-image (x_P,y_P) are given.

Step A: $(x_P,y_P)\to(x,y,z)$. Let us write the equation of the right line corresponding to the line of sight (in system (XYZ)):

$$\frac{x-x_N}{x_A-x_N} = \frac{y-y_N}{y_A-y_N} = \frac{z-z_N}{z_A-z_N}, \tag{3}$$

where (x_N, y_N, z_N) - coordinates of the observer, (x_A, y_A, z_A) - coordinates of a point on the projection plane corresponding to (x_P, y_P). Coordinates of the observer can be obtained like this:

$$\begin{cases} x_N = D\sin l_0 \cos b_0, \\ y_N = D\sin b_0, \\ z_N = D\cos l_0 \cos b_0, \end{cases} \quad (4)$$

where b_0, l_0 – planetocentric coordinates of the point under the observer. Coordinates (x_A,y_A,z_A) can be obtained from (x_P,y_P) by transforming from (X′Y′Z′) to (XYZ) with several consecutive rotations:

$$\begin{cases} x_1 = x_P \cos\psi + y_P \sin\psi; \\ y_1 = -x_P \sin\psi + y_P \cos\psi; \end{cases}$$

$$\begin{cases} x_2 = x_1; \\ y_2 = y_1 \cos\rho - D\sin\rho; \\ z_2 = D - y_1 \sin\rho - D\cos\rho; \end{cases}$$

$$\begin{cases} x_3 = x_2 \cos(P_0 - \psi) + y_2 \sin(P_0 - \psi); \\ y_3 = -x_2 \sin(P_0 - \psi) + y_2 \cos(P_0 - \psi); \\ z_3 = z_2; \end{cases}$$

where P_0 – the position angle of a planet on the image, counted from position "the north is above" anticlockwise;

$$\begin{cases} x_4 = x_3; \\ y_4 = y_3 \cos b_0 + z_3 \sin b_0; \\ z_4 = -y_3 \sin b_0 + z_3 \cos b_0; \end{cases}$$

$$\begin{cases} x_A = x_4 \cos l_0 + z_4 \sin l_0; \\ y_A = y_4; \\ z_A = -x_4 \sin l_0 + z_4 \cos l_0. \end{cases} \quad (5)$$

To calculate coordinates of a point on the surface of the planet, we should solve system (3) together with equation (1), finding unknown x, y, z. We should (if it is necessary) find all solutions – i.e. all points of crossing of the right line with the surface and to choose a point with greatest z.

In a case of ellipsoid (2) problem is reduced to the solving a quadratic equation. In more detail all this is described on our site (see link below).

Step B: (x,y,z)→(b,l). Further, under the definition (b,l) we have:

$$\begin{cases} b = \arcsin\left(y \big/ \sqrt{x^2 + y^2 + z^2} \right), \\ l = \arctan\left(x/z \right) - \pi sign\left(z \right)\left(1 - sign\left(z \right) \right) \big/ 2. \end{cases}$$

Transformation (b,l) → (x_P,y_P): Planetocentric coordinates (b, l) are given.

Step A: (b,l) → (x,y,z). Coordinates of a point on the planet surface (x,y,z) can be found with solving system of the equations:

$$\begin{cases} x\cos l - z\sin l = 0, \\ y\cos b - \left(x\sin l + z\cos l \right)\sin b = 0, \\ F(x,y,z) = 0. \end{cases} \quad (6)$$

In a case of ellipsoid the solution looks like this:

$$\begin{cases} x = r(b,l)\cos b \sin l, \\ y = r(b,l)\sin b, \\ z = r(b,l)\cos b \cos l, \end{cases}$$

where $r(b,l) = 1 \Big/ \sqrt{\dfrac{\cos^2 b \sin^2 l}{A^2} + \dfrac{\sin^2 b}{B^2} + \dfrac{\cos^2 b \cos^2 l}{C^2}}$.

Step B: (x,y,z) → (x_P,y_P). Let us substitute coordinates (x,y,z) to system of the equations (3) and solve this system relatively unknown x_P, y_P.

Transformation (x,y,z) → (α,i,ε): To find the phase angle α, the angles of incidence i and emergence ε, let us use coordinates (x,y,z), obtained in the Step A in any of above mentioned transformations.

Let us obtain a direction of a normal $\bar{N}$ to the surface, differentiating function (1) in point (x,y,z), and then obtain a vector of direction to observer $\bar{P}$:

$$\bar{N} = \left(\frac{dF}{dx}, \frac{dF}{dy}, \frac{dF}{dz} \right), \bar{P} = \left(x_N - x, y_N - y, x_N - z \right),$$

where (x_N, y_N, z_N) - coordinates of the observer (4). The angle of emergence ε we find as an angle between vectors $\bar{N}$ and $\bar{P}$:

$$\cos\varepsilon = \left| \frac{\left(\bar{N}, \bar{P} \right)}{\left\| \bar{N} \right\| \left\| \bar{P} \right\|} \right|,$$

For obtaining the angle of incidence i we should calculate coordinates of the Sun. We know planetocentric coordinates of the point on the surface under the Sun (b_S,l_S) and distance from the Sun to the planet center r_S (as a rule, it is possible to set r_S equal to infinity).

Sun coordinates (x_S,y_S,z_S) can be obtained like (4). And the vector of a direction to the Sun is

$$\bar{S} = \left(x_S - x, y_S - y, z_S - z \right).$$

Then the angle of incidence i and phase angle α can be calculated like this:

$$\cos i = \left| \frac{\left(\bar{N}, \bar{S} \right)}{\left\| \bar{N} \right\| \left\| \bar{S} \right\|} \right|, \quad \cos\alpha = \frac{\left(\bar{P}, \bar{S} \right)}{\left\| \bar{P} \right\| \left\| \bar{S} \right\|}.$$

Conclusion: Thus, using above mentioned transformations of coordinates, it is possible to process images of any objects, shape of which can be presented like (1). The observer and a light source can be placed on arbitrary distance from the planet.

If it is impossible to set the shape of a planet analytically it is necessary to use its discrete representation and then the solution of systems (3)+(1) and (6) should be found numerically.

On our site you can find more detailed deriving of the these formulas, and also a concrete realization of these algorithms for the case of ellipsoidal planet (http://www.univer.kharkov.ua/astron/dslpp/cartography/).